Arithmetical rank of Stanley-Reisner ideals of small arithmetic degree (Algebra, Languages and Computation)

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1 Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring with n variables over a field k with $\deg x_i = 1$ $(i = 1, 2, \dots, n)$. In this article we determine the arithmetical rank of squarefree monomial ideals in R with small arithmetic degree. More precisely, we prove the following theorem:

Theorem. *Let I be a squarefree monomial ideal. Then we have:*

(1)
$$\mathrm{arithdeg} I = \mathrm{reg}\, I \Rightarrow \mathrm{ara}\, I = \mathrm{projdim}\ (R/I).$$

(2)
$$\mathrm{arithdeg} I = \mathrm{indeg} I + 1 \Rightarrow \mathrm{ara}\, I = \mathrm{projdim}\ (R/I).$$

First we fix the terminology we use in this article.

Let I be an ideal of R. We define the arithmetical rank $\mathrm{ara} I$ of I by

$$\mathrm{ara} I := \min\{r;\ \exists a_1, a_2, \dots, a_r \in I \text{ such that } \sqrt{(a_1, a_2, \dots, a_r)} = \sqrt{I}\ \}.$$

In general, $\mathrm{ara} I \geq \mathrm{ht}\, I$. And I is said to be a *set-theoretic complete intersection*, if $\mathrm{ara} I = \mathrm{ht} I$.

Let I be a homogeneous ideal in R and

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow I \to 0$$

a graded minimal free resolution of I over R. Here p is called the *projective dimension* of I over R and denote it by $\mathrm{projdim} I$. We have $\mathrm{projdim}\ (R/I) = \mathrm{projdim} I + 1$. Put $\mu(I) := \sum_j \beta_{0j}$, which stands for the minimum number of generators of I. The *initial degree* indeg I of I and the *relation type* rt(I) of I are defined respectively by

$$\begin{aligned}\operatorname{indeg} I &= \min\{j : \beta_{0j} \neq 0\},\\ \operatorname{rt} I &= \max\{j : \beta_{0j} \neq 0\}.\end{aligned}$$

And the *(Castelnuovo-Mumford) regularity* of I is defined by

$$\operatorname{reg} I = \max\{j - i : \beta_{ij} \neq 0\}.$$

We say that I has *linear resolution* if $\operatorname{reg} I = \operatorname{indeg} I$.

For a simplicial complex Δ on the vertex set $V = \{1, \ldots, n\}$, we mean that Δ is a collection of subsets of V such that

$$F \in \Delta,\ G \subset F \Rightarrow G \in \Delta.$$

We call

$$I_\Delta = (x_{i_1} \cdots x_{i_p};\ i_1 < i_2 < \ldots < i_p,\ \{i_1, \ldots, i_p\} \notin \Delta)$$

the *Stanley–Reisner ideal* of Δ.

Put

$$\Delta^* = \{F \in 2^V : V \setminus F \notin \Delta\},$$

which is also a simplicial complex, and called the *Alexander dual* of Δ. We call I_{Δ^*} the Alexander dual ideal of I_Δ.

2 Arithmetical rank of squarefree monomial ideals

Let $H_I^i(R)$ be the i-th local cohomology module of R with respect to I. The *cohomological dimension* cd I of I is defined to be $\operatorname{cd} I := \max\{i;\ H_I^i(R) \neq 0\}$. It is easy to see $\operatorname{ara} I \geq \operatorname{cd} I$.

When I is a squarefree monomial ideal, the following theorem is known :

Theorem 2.1 (Lyubeznik [Ly1] see also [Te2]). *Let I be a squarefree monomial ideal. Then we have*

$$\operatorname{projdim}\,(R/I) = \operatorname{cd} I.$$

Corollary 2.2. *Let I be a squarefree monomial ideal. Then we have*

$$\text{ara } I \geq \text{projdim } (R/I).$$

In particular, if I is a set-theoretic complete intersection, then R/I is Cohen-Macaulay.

Problem 2.3. Let I be a squarefree monomial ideal. Under what conditions do we have ara I = projdim (R/I)?

We do not always have ara I = projdim (R/I) as the following example shows.

Example 2.4 (Yan [Ya]). Let I be the ideal in $R = k[u, v, w, x, y, z]$ generated by $uvw, uvy, vwx, uwz, uxy, uxz, vxz, vyz, wxy, wyz$. Then I is the Stanley-Reisner ideal of a triangulation of $\mathbf{P}^2(\mathbf{R})$ with six vertices. In this case, ara $I = 4$, which is proved by Yan, using the étale cohomology. On the other hand projdim $(R/I) = 3$ if char $(k) \neq 2$.

We pick up some classes for whose members the equality holds.

Proposition 2.5 ([Te3]). *Let I be a squarefree monomial ideal. If $\mu(I)-$*projdim $(R/I) \leq 1$, *then we have*

$$\text{ara } I = \text{projdim } (R/I).$$

For an ideal I in R, we define the *deviation* $d(I)$ of I by $d(I) = \mu(I) - \text{ht } I$.

Theorem 2.6 ([Te4]). *Let I be a squarefree monomial ideal of deviation 2. Then we have*

$$\text{ara } I = \text{projdim } (R/I).$$

Proposition 2.7. *Let Δ be a disconnected simplicial complex. I.e., let I_Δ be a squarefree monomial ideal with* depth $R/I_\Delta = 1$. *Then we have*

$$\text{ara } I_\Delta = \text{projdim} R/I_\Delta.$$

(*Proof.*) By [Ei-Ev] we have $n - 1 = \text{projdim} R/I_\Delta \leq \text{ara } I_\Delta \leq n - 1$.

Proposition 2.8. *Let Δ be a non-acyclic simplicial complex such that I_Δ has linear resolution. (E.g., I_Δ is a non-Cohen-Macaulay Buchsbaum squarefree monomial ideal with linear resolution.) Then we have*

$$\text{ara } I_\Delta = \text{projdim} R/I_\Delta.$$

(*Proof.*) By [Gr] we have $n - \text{indeg} I_\Delta + 1 = \text{projdim} R/I_\Delta \leq \text{ara } I_\Delta \leq n - \text{indeg} I_\Delta + 1$.

3 Squarefree monomial ideals of small arithmetic degree

We define the *arithmetic degree* arithdeg I of a squarefree monomial ideal I by

$$\operatorname{arithdeg} I = \sharp(\operatorname{Ass} R/I).$$

For squarefree monomial ideals, we have the following relations:

Theorem 3.1 (Hoa-Trung[Ho-Tr], Stückrad, Frübis-Terai[Fr-Te]). *Let I be a squarefree monomial ideal. Then we have*

$$\operatorname{indeg} I \leq \operatorname{reg} I \leq \operatorname{arithdeg} I.$$

The arithmetical rank is known when the arithmetic degree agrees with the initial degree:

Theorem 3.2 (Schenzel-Vogel[Sche-Vo], Schmitt-Vogel[Schm-Vo]). *If a squarefree monomial ideal I satisfies* $\operatorname{arithdeg} I = \operatorname{indeg} I$, *then after a suitable change of variables, I is of the form*

$$I = (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \ldots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}),$$

and $\operatorname{projdim}(R/I) = \sum_{i=1}^{q} j_i - q + 1$.

Put $a_\ell = \sum_{\ell_1+\ell_2+\cdots+\ell_q=\ell} x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q}$ *for* $\ell = q, q+1, \ldots, \sum_{i=1}^{q} j_i$. *Then we have* $\sqrt{(a_\ell;\ \ell = q, q+1, \ldots, \sum_{i=1}^{q} j_i)} = I$.

Hence $\operatorname{ara} I = \operatorname{projdim}\ (R/I)$.

Now we consider the case that the arithmetic degree is equal to regularity:

Theorem 3.3. *Let I be a squarefree monomial ideal with* $\operatorname{arithdeg} I = \operatorname{reg} I$. *Then we have*

$$\operatorname{ara} I = \operatorname{projdim}\ (R/I).$$

To prove the above theorem we define the *size* of a monomial ideal I, which is introduced by Lyubeznik. Let $I = \cap_{j=1}^{r} Q_j$ be an irredundant primary decomposition of I, where the Q_i are monomial primary ideals. Let h be the height of $\sum_{j=1}^{r} Q_j$, and denote by v the minimum number t such that there exist $j_1, \ldots, j_t$ with $\sqrt{\sum_{i=1}^{t} Q_{j_i}} = \sqrt{\sum_{j=1}^{r} Q_j}$. Then size$I = v + (n - h) - 1$. Then we have:

Lemma 3.4 (Lyubeznik[Ly2]). *Let I be a (squarefree) monomial ideal in R. Then* ara$I \leq n -$ size$I$.

The form is determined for a squarefree monomial ideal I with arithdeg$I$ = reg I as follows:

Lemma 3.5 (Hoa-Trung[Ho-Tr]). *Let I be a squarefree monomial ideal in R such that* arithdeg$I$ = reg I. *Then after a suitable change of variables, I is of the form*

$$I = (y_1, x_{i_{11}}, x_{i_{12}}, \ldots, x_{i_{1j_1}}) \cap (y_2, x_{i_{21}}, x_{i_{22}}, \ldots, x_{i_{2j_2}}) \cap \ldots \cap (y_q, x_{i_{q1}}, x_{i_{q2}}, \ldots, x_{i_{qj_q}}),$$

and

$$\mathrm{projdim}(R/I) = \deg \mathrm{lcm}(x_{i_{11}}, x_{i_{12}}, \ldots, x_{i_{1j_1}}, x_{i_{21}}, x_{i_{22}}, \ldots, x_{i_{2j_2}}, \ldots, x_{i_{q1}}, x_{i_{q2}}, \ldots, x_{i_{qj_q}}) + 1.$$

Lemma 3.6. *Let I be a squarefree monomial ideal in R such that* arithdeg$I$ = reg I. *Then we have*

$$\mathrm{projdim}\ (R/I) = n - \mathrm{size}I.$$

(*Proof.*) We may assume that every variable is zero divisor on R/I. Since size$I + 1$ = arithdeg$I$ = reg I by the above lemma, it is enough to prove to

$$\mathrm{projdim}\ (R/I) + \mathrm{reg}I = n + 1.$$

Let J be the Alexander daul ideal of I. Then we have

$$J = (y_1 x_{i_{11}} x_{i_{12}} \cdots x_{i_{1j_1}}, y_2 x_{i_{21}} x_{i_{22}} \cdots x_{i_{2j_2}}, \ldots, y_q x_{i_{q1}} x_{i_{q2}} \cdots x_{i_{qj_q}}).$$

Since projdim (R/I) = reg$J$ and reg$I$ = projdim (R/J) (see [Te1]), it is enough to prove

$$\mathrm{projdim}\ (R/J) + \mathrm{reg}J = n + 1.$$

Because of the form of the ideal J, the Taylor resolution of J gives a minimal free resolution of J. Hence the last syzygy determines the regularity. Since every variable is zero divisor on R/J, reg$J = n -$ projdim$J = n -$ projdim$(R/J) + 1$. QED

Now Theorem 3.3 is clear by Lemmas 3.4 and 3.6.

Next we consider a squarefree monomial ideal whose arithmetic degree is one bigger than its initial degree:

Theorem 3.7. *Let I be a squarefree monomial ideal with* $\mathrm{arithdeg} I = \mathrm{indeg} I + 1$. *Then we have*

$$\mathrm{ara}\, I = \mathrm{projdim}\ (R/I).$$

To prove the above theorem we use:

Lemma 3.8. *Let I be a squarefree monomial ideal with* $\mathrm{arithdeg} I = \mathrm{indeg} I + 1$. *Then I is one of the following forms after a suitable change of the variables:*

(1)

$$\begin{aligned} I = {} & (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \ldots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ & \cap (x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $q \geq p \geq 2$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_q \geq 1$.

(2)

$$\begin{aligned} I = {} & (x_{11}, x_{12}, \ldots, x_{1j_1}) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}) \cap \ldots \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \\ & \cap (x_{q+1,1}, x_{q+1,2}, \ldots, x_{q+1,j_{q+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pi_p}), \end{aligned}$$

where $q \geq p \geq 1$, $1 \leq i_\ell < j_\ell$ $(\ell = 1, 2, \ldots, p)$, $j_{p+1}, \ldots, j_q, j_{q+1} \geq 1$.

(3)

$$\begin{aligned} I = {} & (x_{11}, x_{12}, \ldots, x_{1j_1}, y_1, \ldots, y_p) \cap (x_{21}, x_{22}, \ldots, x_{2j_2}, y_1, \ldots, y_p) \cap (x_{31}, x_{32}, \ldots, x_{3j_3}) \cap \cdots \\ & \cap (x_{q1}, x_{q2}, \ldots, x_{qj_q}) \cap (x_{q+1,1}, x_{q+1,2}, \ldots, x_{q+1,j_{q+1}}, x_{11}, x_{12}, \ldots, x_{1i_1}, x_{21}, x_{22}, \ldots, x_{2i_2}), \end{aligned}$$

where $q \geq 2$, $p \geq 1$, $1 \leq i_\ell \leq j_\ell$ $(\ell = 1, 2)$, $j_3, \ldots, j_q \geq 1$, $j_{q+1} \geq 0$.

(Proof.) Let I be a squarefree monomial ideal with $\mathrm{arithdeg} I = \mathrm{indeg} I + 1$, and J its Alexander dual ideal. Then J satisfies that $\mu(J) = \mathrm{ht} J + 1$, that is J is an almost complete intersection. Such J are classified in [Te3]. QED

(Proof of Theorem 3.7.) We check the equality for all the cases in the above lemma. Let J be the Alexander dual ideal of I.

(1)We may assume that $j_1 - i_1 = \min\{j_\ell - i_\ell;\ \ell = 1, 2, \ldots, p\}$. Then

$$\mathrm{projdim}\ (R/I) = \mathrm{reg} J = i_1 + j_2 + \cdots + j_q - q + 1.$$

Put $a_\ell = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_q=\ell \\ \ell_1\le i_1 \text{ or } \ell_2\le i_2 \text{ or } \ldots \text{ or } \ell_p\le i_p}} x_{1\ell_1}x_{2\ell_2}\cdots x_{q\ell_q}$ for $\ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t$. Then we have $\sqrt{(a_\ell;\ \ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t)} = I$ by [Schm-Vo, Lemma]. Hence ara I = projdim (R/I).

(2)By Theorem 3.3 the equality holds in this case.

(3) (i) The case of $j_{q+1} > 0$. By Theorem 3.3 the equality holds.

(ii) The case of $j_{q+1} = 0$ and $i_\ell < j_\ell$ ($\ell = 1, 2$). We may assume that $j_1 - i_1 \le j_2 - i_2$. Then

$$\text{projdim } (R/I) = \text{reg} J = i_1 + j_2 + \cdots + j_q - q + 1 + p.$$

For simplicity, we mean that $x_{1j_1+i} = y_i$ and $x_{2j_2+i} = y_i$ for $i = 1, 2, \ldots, p$.

Put $a_\ell = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_q=\ell, \\ \ell_1\le i_1 \text{ or } \ell_2\le i_2}} x_{1\ell_1}x_{2\ell_2}\cdots x_{q\ell_q}$ for $\ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t + p$. Then we have $\sqrt{(a_\ell;\ \ell = q, q+1, \ldots, i_1 + \sum_{t=2}^{q} j_t + p)} = I$ by [Schm-Vo, Lemma]. Hence ara I = projdim (R/I).

(iii) The case of $j_{q+1} = 0$ and ($i_1 = j_1$ or $i_2 = j_2$). We may assume that every variable is a zero divisor on R/I. Then R/J is Cohen-Macaulay with $a(R/J) = 0$. Hence by Proposition 2.8 the equality holds in this case. QED

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